ON COMMUTATIVITY OF RA-SEMIGROUPS

D.Leelavardhini[1,2], **Dr. G. Shobha Latha**[3]

1Research scholar, Department of Mathematics, S K University,

Ananthapurum, Andhra Pradesh, India.

2 Lecturer, Department of of Mathematics, STSN Govt Degree College,

Kadiri, Ananthapuram Dist, Andhra Pradesh, India

3 Professor, Department of Mathematics, S K University,

Ananthapurum, Andhra Pradesh, India.

ABSTRACT:

An algebraic structure midway between a groupoid and commutative semigroup appeared in1972. M.A.Kazim and MD.Naseerudin introduced left almost semigroups as a generalisation of commutative semigroups. They have introduced the braces on the left of the ternary commutative law $abc = cba$ to get a new pseudo associative law, i.e., $(ab)c = (cb)a$. It is since than called left invertive law. A groupoid satisifying the left invertive law is called a left almost semigroup and is abbreviated as LA- semigroup. Similarly , groupoid satisfying the right invertive law,$a(bc) = c(ba)$ is called a rightt almost semigroup and is abbreviated as RA- semigroup

In this paper we will prove some of the properties of RA-semigroups. . In this paper in section 2 we define RA-semigroup with examples and prove some of the basic results. In section 3 we study the properties of commutative and bi-commutative and transitively commutative RA-semigroups. In 4 we prove some of the properties of the anti-commutative RA-semigroups.

Keywords: *RA-semigroup, Commutative RA-semigroups, Bi-commutative RA-semigroup, Anti-commutative RA-semigroup.*

1 INTRODUCTION :

The algebraic object encountered in this chapter is a set G with a binary operation ' . ' satisfying right invertive law is same as the algebraic structure " Right almost semigroup i.e., RA-semigroup" defind by MD.Naseeruddin in his Ph.D theses with the title "Some studies on almost semigroups and flocks" . He defined RA-semigroup as a groupoid satisfying right invertive law. i.e.,

$$a(bc) = c(ba) \ \forall a, b ,c, \in G$$

Definition : Let G be a non empty set and ' · ' be a binary operation from GxG →G. Then (G,.) is called an RA-semigroup if it satisfies,

$$a(bc) = c(ba) \quad \forall\ a, b ,c, \in G$$

The following multiplication table shows the existence of an RA-semigroup.

.	x	y	z
x	x	z	y
y	y	x	z
z	z	y	x

Definition : An RA-semigroup is called a commutative RA-semigroup if $ab = ba \ \forall\ a, b \in G$

Definition : Bi-Commutative RA-semigroup (BC- RA-semigroup) :

An RA-semigroup G is called right commutative RA-semigroup (RC- RA-semigroup) if $a(bc) = a(cb)$, for all $\forall\ a, b ,c, \in G$.

An RA-semigroup G is called a left commutative RA-semigroup (LC- RA-semigroup) if $(ab)c = (ba)c$, for all $\forall\ a, b ,c, \in G$

An RA-semigroup G is called a bi-commutative RA-semigroup (BC- RA-semigroup) if it is both LC- RA-semigroup and RC- RA-semigroup.

Definition : Anti-commutative RA-semigroup**:** An RA-semigroup G is called anti-commutative RA-semigroup if the identity , $ab = ba \implies a = b$ holds $\forall\ a, b \in G$

Alternative RA-semigroup : An RA-semigroup G is called left alternative RA-semigroup if it satisfies the identity , $(aa)b = a(ab)$. for all $a,b \in G$

An RA-semigroup is called right alternative RA-semigroup G if it satisfies the identity, $(ab)b = a(bb)$, for all $a,b \in G$

Self-Dual RA-semigroup : An RA-semigroup which satisfies left invertive law $(ab)c = c(ba)$ for all $a,b,c \in G$ is called Self -Dual RA-semigroup.

Nuclear square : An RA-semigroup G is called left nuclear square if $\forall a,b,c, \in G$, $a^2(bc) = (a^2b)c$. Similarly S is called right nuclear square if $\forall a,b,c, \in G$, $(ab)c^2 = a(bc^2)$ and middle **nuclear square if $\forall a,b,c, \in G$, $(ab^2)c = a(b^2c)$.**

Right transitive RA-semigroup : An RA-semigroup G is called right transitive if , $ab.cb = ac$ for all $a,b, c \in G$

Left transitive RA-semigroup : An RA-semigroup G is called left transitive if , $ab.ac = bc$ for all $a,b, c \in G$

Locally associative: An RA-semiroup G is said be locally associative if, $a^2a = aa^2 \quad \forall \ a \in G$

Cancellative RA-semigroup : Let G be an RA-semigroup. If for all $a, b ,c, \in G$, $ab = ac \implies b =c$ then we say that G is a left cancellative RA-semigroup

Let G be an RA-semigroup. If for all $a, b ,c, \in G$, $ba = ca \implies b =c$ then we say that G is a right cancellative RA-semigroup

Let G be an RA-semigroup. If for all $a, b ,c, \in G$, $ab = ac \implies b =c$, and also $ba = ca \implies b = c$, then we say that G is a cancellative RA-semigroup

Let G be an RA-semigroup. If for all $a, b ,c, \in G$, $ab = ca \implies b =c$, and also $ba = ac \implies b = c$, then we say that G is a cross-cancellative RA-semigroup

2: SOME BASIC RESULTS ON RA-SEMIGROUPS.

2.1 Lemma : An RA-semigroup G satisfies medial law.

i.e., $(ab)(cd) = (ac)(bd) \quad \forall a, b, c \in G$.

Proof : Using right invertive law,

$(ab)(cd) = d(c(ab)) = d(b(ac) = (ac)(bd)$

2.2 Lemma : If right identity 'e' exists in RA-semigroup then it is unique.

Proof : If possible there exists another right identity say f, then

$f=fe$ and $ef=e$ and $f=fe=f(ee)=e(ef)=ee=e \implies f = e$

2.3 Lemma : Ia an RA-semigroup with right identity, paramedial law holds.

i.e., $(ab)(cd) = (db)(ca)$

Proof : $(ab)(cd) = (ab)((cd)e) =e((cd)(ab)) =e(b(a(cd))) = e(b(d(ca))) = e((ca)(db))$

$=(db)((ca)e)= (db)(ca)$.

2.4 Lemma : In an RA-semigroup with right identity e ,

$ab=cd \iff ba=dc$

Proof : (i) $ab = cd \implies ba = d$

$$ba = b(ae) = e(ab) = e(cd) = d(ce) = dc.$$

Similarly we can show that $(ba) = (dc) \implies (ab) = (cd)$

2.5 Lemma : If an RA-semigroup G contains a right identity the following law holds $(ab)c = (ac)b, . \forall a, b, c \in G$.

Proof : $(ab)c = (ab)(ce) = e\big(c(ab)\big) = e\big(b(ac)\big) = (ac)(be) = (ac)b$.

3: ON COMMUTATIVITY OF RA-SEMIGROUPS

3.1 Theorem : An RA-semigroup G is a commutative semigroup iff the following law holds $\forall\, a, b, c \in$ G.

$a(bc)=(ba)c$ ---- (a)

Proof : (i) Let the given condition (a) holds in G,

and $a(bc) = c(ba)$ ---- (b) (right invertive law)

from (a) and (b), we have

$(ba)c = c(ba) \Longrightarrow$ G is commutative.

and $a(bc) = (ba)c = (ab)c \Longrightarrow$ Gis associative.

∴ G is a commutative semigroup.

(ii) Let G be a commutative semigroup.

then, $a(bc) = (ab) = c(ab) = c(ba) = (ba)c \Longrightarrow a(bc) = (ba)c$

3.2 Theorem : An RA-semigroup with left identity is a commutative semigroup.

Proof : Let G is an RA-semigroup with left identity e .

then, $ea = a$

$ab = a(eb) = b(ea) = ba \Longrightarrow ab = ba \Longrightarrow$ G is commutative.

$a(bc) = c(ba) = c(ab) = (ab)c$ (Since G is commutative)

$a(bc) = (ab)c \Longrightarrow$ G is associative

∴ G is a commutative semigroup.

3.3 Theorem: Let G be a RA-semigroup with right identity. If G is left alternative then G is a commutative semigroup.

Proof: G is an RA-semigroup

Let G is left alternative then,

$aa.b = a.ab \quad \forall\, a, b \in$ G.

$aa.b = a.ab = b.aa \Longrightarrow a^2.b = b.a^2$

Now replace a by e we have, $e^2.a = a.e^2 \Longrightarrow a.e = e.a$

Since e is the right identity in G we have $a.e = e.a = a \Longrightarrow e$ is the identity in G

Now by right invertive law and identity in G we have,

$ab = a(be) = e(ba) = ba \Longrightarrow$ G is commutative

and by commutativity and right invertive law we have,

$a(bc) = c(ba) = c(ab) = (ab)c \Longrightarrow$ G is associative

∴ G is a commutative semigroup

3.4 Theorem : A commutative RA-semigroup is,

(i) Associative

(ii) Permutable

(iii Self-Dual

(iv) Bi-commutative

(v) Alternative

(vi) Paramedial

(vii) Nuclear square

Proof: Let G is a commutative RA-semigroup

(i) Consider $a(bc) = c(ba) = c(ab) = (ab)c \implies a(bc) = (ab)c \implies$ G is associative

(ii) Let G be a commutative RA-semigroup, then

Consider $a(bc) = c(ba) = c(ab) = b(ac) = (ac)b \implies a(bc) = b(ac) \implies$ Gis right permutable

similarly $a(bc) = c(ba) = c(ab) = b(ac)$ i.e., Gis left permutable.

(iii) Consider $a(bc) = c(ba)$

By using commutativity on both sides we get,

$$a(bc) = c(ba) \Leftrightarrow (bc)a = c(ab) \Leftrightarrow (cb)a = (ab)c$$

Right invertive law ⇔ Left invertive law. ⟹G is Self-Dual RA-semigroup.

(iv) Consider $a(bc)$ & $(ab)c$

Since G is commutative ,we have

$a(bc) = a(cb)$ (commutativity)

$(ab)c = (ba)c$ (commutativity)

$\implies$ G is Bi-commutative.

(v) $a(ab) = b(aa) = (aa)b \implies$ G is left alternative.

$a(bb) = b(ba) = b(ab) = (ab)b \implies$ G is right alternative

$\implies$ G is an alternative RA-semigroup.

(vi) Consider $(ab)(cd) = (cd)(ab)$ (commutativity)

$= (dc)(ba)$ (commutativity)

$= (db)(ca)$ (medial law)

$\implies$ G is paramedial

(vii) since Commutativity ⟹ Associativity we have,

$a^2(bc) = (a^2b)c,\ (ab)c^2 = a(bc^2),\ (ab^2)c = a(b^2c).\ \forall a, b, c, \in G \implies$G is a nuclear square.

3.5 Theorem : Let G be an RA-semigroup. Then G is a commutative semigroup if G satisfies any one of the following.

(i) G is slim RA-semigroup

(ii) G is left alternative

(iii) G is right alternative satisfying cross-cancellation

(iv) Idempotent and paramedial

(v) Left commutative with right cancellation

(vi) Right commutative with left cancellation

(vii) Self - dual with right identity

(viii) Left transitive

(ix) Right transitive

Proof: Let G be an RA-semigroup and let $a, b, c, \in$ G

(i) Let G be a slim RA-semigroup. Then, $a(bc) = ac$

Consider $ab = a(bb) = b(ba) = ab \Longrightarrow ab = ba \Longrightarrow$ G is commutative.

(ii) Let G be a left alternative RA-semigroup

By left alternativity in G we have , $(aa)b = a(ab) \quad \forall a, b \in$ G.

Using right invertive law on the right we get $(aa)b = b(aa) \quad \forall a, b \in$ G.

$\Longrightarrow$ G is commutative.

(iii) Let G is right alternative RA-semigroup satisfying cross-cancellation

Since G is right alternative we have , $(ab)b = a(bb) \quad \forall a, b \in$ G.

Using right invertive law on the right we get $(ab)b = b(ba)$

using cross-cancellativity in G we have $ab = ba \quad \forall a, b \in$ G $\Longrightarrow$ G is commutative.

(iv) Let G be an idempotent RA-semigroup. Then $a^2 = a \quad \forall a, \in$ G.

Let G be paramedial then, $(ab)(cd) = (db)(ca)$

Consider $ab = (ab)^2 = (ab)(ab) = (bb)(aa) = b^2a^2 = ba \Longrightarrow ab = ba \Longrightarrow$ G is commutative

(v) Let G be a left commutative Ra-semigroup with right cancellativity

G is left commutative we have , $(ab)c = (ba)c \quad \forall a, b, c \in$ G.

cancellativity in G we have $ab = ba \quad \forall a, b \in$ G $\Longrightarrow$ G is commutative

(vi) Let G be a right commutative using right RA-semigroup.

Since G is right commutative we have , $a(bc) = a(cb) \quad \forall a, b, c \in$ G.

using left cancellativity in G we have $bc = cb \Longrightarrow$ G is commutative.

A(vii) Let G be an RA-semigroup with right identity. Then $ae = a \quad \forall a, \in$ G.

Let G be self-dual then, $(ab)c = (cb)a$

Consider $ea = (ee)a = (ae)e = ae = a \Longrightarrow eaa \Longrightarrow e$ is the left identity $\Longrightarrow e$ is the identity

Now $ab = a(be) = e(ba) = ba \Longrightarrow ab = ba \Longrightarrow$ G is commutative

(viii) Let G be a left transitive RA-semigroup

By left transitive condition in G we have , $bc = bb.bc$

$bb.bc = bb(bb.bc) \qquad (\text{Since } bc=bb.bc).$

Using right invertive law on the right we get

$bc = bb(bb.bc) = bc(bb.bb) = bc.bb = cb \qquad (\text{by left transitivity in G})$

Thus $bc = cb \Longrightarrow$ G is commutative.

(ix) Let G be a right transitive RA-semigroup

By right transitive condition in G we have , $ac = ac.aa = (a.ac)(a.aa)$

Using right invertive law on the right we get $ac = (c.aa)(a.aa) \quad (\text{by right transitivity in G})$

$= ca \quad (\text{by right transitivity in G})$

Thus $ac = ca \Longrightarrow$ G is commutative.

In RA-semigroups always commutativity implies associarivity.

Hence in all the above cases G is associative and hence G is a commutative semigroup.

3.6 Theorem: Let G be a left-cancellative RA-semigroup. Then G is transitively commutative.

Proof: Let G be an RA-semigroup. Let $a, b, c \in G$ such that $ab = ba$ and $bc = cb$.

By right invertive law in G we have,

$$a(bc) = c(ba)$$

Since $ab = ba$ and $bc = cb$ we have,

$$a(cb) = c(ab) \Longrightarrow b(ca) = b(ac)$$

using left-cancellation we get, $ca = ac$

$$ab = ba \text{ and } bc = cb \Longrightarrow ca = ac \Longrightarrow \text{G is transitively commutative}$$

3.7 Theorem: Let G be an RA- semigroup. Then G is transitively commutative if ,

(i) G is left transitive

(ii) G is right transitive

Proof:

(i) Let G be a left transitive RA-semigroup $\Longrightarrow ac = ba.bc \quad \forall\, a, b, c \in$ G

Let $ab = ba$ and $bc = cb$.

Now we use the above assumptions and right invertive law to show that $ac = ca$

Consider $ac = ba.bc$

$= ba.cb$ $(bc=cb)$

$= b(c.ba)$ (right invertive law)

$= b(a.bc))$ (right invertive law)

$= bc.ab)$ (right invertive law)

$=bc.ba$ $(ab = ba)$

$=ca$ (G is left transitive)

$ab =ba \;\&\; bc =cb \Longrightarrow ac = ca \quad \forall\, a, b, c \in$ G

$\Longrightarrow$ G is transitively commutative

(ii) Let G be a right transitive RA-semigroup $\Longrightarrow ac = ab.cb \quad \forall\, a, b, c \in$ G

Let $ab = ba$ and $bc = cb$.

Now we use the above assumptions and right invertive law to show that $ac = ca$

Consider $ac = ab.cb$

$= ba.cb$ $(bc=cb)$

$= b(c.ba)$ (right invertive law)

$= b(a.bc))$ (right invertive law)

$= b(a.cb)$ $(bc = cb)$

$=cb.ab$ (right invertive law)

$=ca$ (G is left transitive)

$ab =ba \;\&\; bc =cb \Longrightarrow ac = ca \quad \forall\, a, b, c \in$ G $\Longrightarrow$ G is transitively commutative

3.8 Theorem: Let G be a RA- semigroup with the condition $a(bc)=ac$ for all a, b, c in G. Then G is,

(i) Left commutative

(ii) Right commutative

(iii) Transitively commutative

Proof: Let G be an RA-semigroup

And let $a, b, c \in G$ such that $a(bc)=ac$

(i) To show that G is left commutative , we have to show $(ab)c = (ba)c$

For this consider $(ab)c$

Using right invertive law, medial law and slim groupoid property we show that G is left commutative.

$(ab)c = (ab)(bb(c)) = c(bb.ab) = c(ba.bb) = c(b(b.ba) = c(b(ba)) = (ba)(bc) = (ba)c$

$(ab)c = (ba)c\ ) \implies$ G is a left commutative RA-semigroup

(ii) To show that G is right commutative , we have to show $a(bc) = a(cb)$

Using right invertive law, medial law and slim groupoid property we show that G is right commutative.

$a(bc) = a(cc(bc)) = a(cb.cc) = a(c(c.cb)) = a(c(cb)) = a(cb)$

$a(bc) = a(cb) \implies$ G is a right commutative RA-semigroup

(iii) Let $a, b, c \in G$ such that $a(bc)=ac$

and let $ab = ba$ & $bc = cb$

Consider $ac = a(bc) = a(b(bc)) = a(c(bb)) = a(bb) = ab$ ----(1)

And $ab = ba = b(ca) = b(c(ca)) = b(a(cc) = b(cc) = bc$ ------(2)

Again $bc = cb = c(ab) = c(a(ab) = c(b(aa)) = c(aa) = ca$ ------(3)

From (1), (2) & (3) $ac = ab = ba = bc = cb = ca \implies ac = ca$

$\implies$ G is transitively commutative RA-semigroup

3.9 Theorem: Let G be an RA-semigroup with the identity $a(bc) = (ac)b \ \forall a, b, c \in$ G.Then G is left commutative

Proof: G is an RA-semigroup with the identity $a(bc) = (ac)b \ \forall a, b, c \in$ G.

Consider $(ab)c = a(cb)$ (by the assumption $a(bc) = (ac)b$)

$= b(ca)$ (right invertive law)

$= (ba)c$ (by the assumption $a(bc) = (ac)b$)

$(ab)c = (ba)c \quad \forall a, b, c \in$ G $\implies$ G is left commutative

3.10 Theorem: Let G be (right commutative) RC-RA-semigroup. Then G is a commutative semigroup if,

(i) G has right identity

(ii) G is cancellative

Proof: G is a right commutative RA-semigroup

(i) Let e be the right identity in G

Consider $ab = a(be) = a(eb) = b(ea) = b(ae) = ba \implies$ G is commutative

G is commutative $\implies$ G is associative $\implies$ G is commutative semigroup.

(ii) G is right commutative $\implies a(bc) = a(cb) \ \forall a, b, c \in$ G.

by left cancellativity we have , $bc = cb \Longrightarrow$ G is commutative

G is commutative $\Longrightarrow$ G is associative $\Longrightarrow$ G is commutative semigroup.

4 ANTI-COMMUTATIVITY OF RA-SEMIGROUPS.

4.1 Theorem : Let G be an anti-commutative RA-semigroup with right identity 'e ' . Then ,

(i) G is quasi-cancellative.

(ii) G is unipotent

(iii) G is RA-3-band

Proof:

Let G be an anti-commutative RA-semigroup with right identity. Then, $\forall\ a, b, c \in G$ we have

$$a(bc) = c(ba) \qquad \text{------}(I)$$

$$ab = ba \Longrightarrow a=b \qquad \text{------}(II)$$

$$(ab)c = (ac)b \qquad \text{-------}(III)$$

$$ab = cd \Longrightarrow ba = dc \qquad \text{-----}\ (IV)$$

(i) Let $a^2 = ab \Longrightarrow aa=ab \Longrightarrow aa=ba \Longrightarrow ab = ba \Longrightarrow a=b$ (Since G is anti-commutative)

$$a^2 = ab \Longrightarrow a=b$$

Similarly let, $b^2=ba \Longrightarrow bb=ba \Longrightarrow bb=ab \Longrightarrow ba=ab \Longrightarrow a = b$ (Since G is anti-commutative)

$$b^2 = ba \Longrightarrow a=b$$

Hence G is quasi-cancellative

(ii) consider $a^2b^2 = aa.bb$

By medial law $a^2b^2 = ab.ab$

By (iv) $b^2a^2 = ab.ab = aa.bb = a^2b^2$

By anti-commutativity in G $b^2a^2 = a^2b^2 \Longrightarrow a^2=b^2 \Longrightarrow$ G is unipotent

(iii) Consider $a(a.aa)$

by right invertive law $a(a.aa) = aa.aa$

by (IV) $(a.aa)a = aa.aa$

again using right invertive law on the right, $(a.aa)a = a(a.aa)$

from anti-commutativity property $a.aa = a \Longrightarrow$ G is RA-3-band.

4.2 Theorem : Let G be an anti-commutative RA-semigroup . Then ,

G is transitively commutative.

Proof: Let G be an anti-commutative RA-semigroup

And let $a, b, c \in$ G such that $ab = ba$ & $bc = cb$

Now we have to show that $ac = ca$

Since G is anti-commutative, $ab = ba \Longrightarrow a = b$

and $bc = cb \Longrightarrow b = c$

$$a = b \ \&\ b = c \Longrightarrow a = c \Longrightarrow ac = ca$$

$ab = ba$ & $bc = cb \Longrightarrow ac = ca \Longrightarrow$ G is transitively commutative.

G is anti-commutative $\Longrightarrow$ G is transitively commutative

4.3 Theorem : Let G be an anti-commutative RA-semigroup . Then G is an RA-semigroup band if and only if G is locally associative.

Proof:

(i) Let G be a RA-semigroup band

then $a^2 = a \quad \forall \; a \in G$

clearly we have $a^2a = aa^2 \Longrightarrow$ G is locally associative

(ii) Let G be locally associative

then $a^2a = aa^2 \quad \forall \; a \in G$

By anti- commutativity in G we have $a^2 = a \quad \Longrightarrow$ G is an RA-semigroup band

4.4 Theorem : Let G be an anti-commutative RA-semigroup. If G is paramedial then G is,

(i) Unipotent

(ii) Rectangular

Proof: Let G be an anti-commutative paramedial RA-semigroup

(i) Consider $a^2b^2 = aa.bb = ab.ab$ (medial law)

$=bb.aa$ (paramedial law)

$= b^2a^2$

$a^2b^2 = b^2a^2$

By anti-commutativity of G , $a^2b^2 = b^2a^2 \;\Longrightarrow\; a^2 = b^2$

$\Longrightarrow$ G is unipotent.

(ii) Consider $(ab.ad)(cb.cd) = (aa.bd)(cc.bd)$ (medial law)

$= (aa.cc)(bd.bd)$ (medial law)

$= (ca.ca)(bd.bd)$ (paramedial law)

$= (cc.aa)(bd.bd)$ (medial law)

$= (cc.bd)(aa.bd)$ (medial law)

$= (cb.cd)(ab.ad)$ (medial law)

$\Longrightarrow (ab.ad)(cb.cd) = (cb.cd)(ab.ad)$

By anti-commutativity of G $(ab.ad)(cb.cd) = (cb.cd)(ab.ad) \Longrightarrow$ G is rectangular

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Right Almost semigroups with conjugates

D.Leelavardhini

Research scholar, Department of Mathematics, S K University,

and

Lecturer, Department of of Mathematics, STSN Govt Degree College, Kadiri,

Ananthapuram Dist, Andhra Pradesh, India

Dr. G. Shobha Latha

Professor, Department of Mathematics, S K University,

Ananthapurum, Andhra Pradesh, India.

ABSTRACT : An algebraic structure midway between a groupoid and commutative semigroup appeared in1972. M.A.Kazim and MD.Naseerudin introduced left almost semigroups as a generalisation of commutative semigroups. They have introduced the braces on the left of the ternary commutative law $abc = cba$ to get a new pseudo associative law, i.e., $(ab)c = (cb)a$. It is since than called left invertive law. A groupoid satisifying the left invertive law is called a left almost semigroup and is abbreviated as LA- semigroup. Similarly , groupoid satisfying the right invertive law,$a(bc) = c(ba)$ is called a rightt almost semigroup and is abbreviated as RA- semigroup.

G.I.MOGHADDAM and .PADMANABHAN in their paper "cancellative semigroups admitting conjugates" considered cancellative semigroup and conjugate of elements. Based on this in this paper we study some equational classes of RA-semigroups admitting conjugates.

Key words: ***RA-semigroup , cancellative semigroup, admits conjugates.***

I.INTRODUCTION

While every subsemigroup of a group is cancellative, a famous theorem of A.I.Mal'cev shows that not every cancellative semigroup is embeddable in a group. In fact, Mal'cev gave an infinite set of necessary and sufficient conditions for the possibility of imbedding a cancellation semigroup into a group and proved that no finite set of such conditions would suffice. This calls for a finite set of sufficient conditions that would guarantee the group embeddability. Patterned after the classical quotient construction, Oystein Ore discovered the "principle of common left multiple" to embed a non-commutative domain into a division ring.Using this as backdrop, Mal'cev, B.H.Neumann and Taylor developed semigroup equivalents of nilpotent groups of class n and proved that the cancellative semigroups of nilpotent class n are embeddable in groups of same nilpotency class. After that G.I.MOGHADDAM, and R.PADMANABHAN studied some equtional classes of enriched semigroups i.e., semigroups admitting conjugates and proved that all the valid group theory implications do carry over to the equational theory of semigroups admitting conjugates.

G.I.MOGHADDAM and .PADMANABHAN in their paper "cancellative semigroups admitting conjugates" considered cancellative semigroup and conjugate of elements. Based on this in this paper we study some equational classes of RA-semigroups admitting conjugates.

II. PRELIMINARIES

RA-semigroup: A groupoid $(G,.)$ satisfying the right invertive law , $a(bc) = c(ba)$ $\forall\ a, b, c \in$ G is called an RA-semigroup

L-cyclic groupoid : A groupoid G is called L-cyclic if , $a(bc) = c(ab) = b(ca)$ $\forall\ a, b, c \in$G.

R-cyclic groupoid : A groupoid G is called R-cyclic if , $(ab)c = (ca)b = (bc)a$ $\forall\ a, b, c \in$ G.

L-cyclic-RA-semigroup : An RA-semigroup is called a left cyclic (L-cyclic) , if $a(bc) = c(ab) = b(ca)$ $\forall\ a, b, c \in$ G.

R-cyclic-RA semigroup : An RA-semigroup is called a right cyclic (R-cyclic) , if $(ab)c = (ca)b = (bc)a$ $\forall\ a, b, c \in$ G.

RC-RA-semigroup: An RA-semigroup G is called right commutative RA-semigroup (RC- RA-semigroup) if for all a,b,c $\in$ G, a(bc) = a(cb).

LC-RA-semigroup : An RA-semigroup G is called a left commutative RA-semigroup (LC- RA-semigroup) if for all a,b,c $\in$ G, (ab)c = (ba)c.

An RA-semigroup G is called a bi-commutative RA-semigroup (BC- RA-semigroup) if it is both LC- RA-semigroup and RC- RA-semigroup.

Alternative RA-semigroup : An RA-semigroup G is called left alternative RA-semigroup if it satisfies the identity , (aa)b = a(ab). for all a,b $\in$ G

An RA-semigroup is called right alternative RA-semigroup G if it satisfies the identity, $(ab)b = a(bb)$, for all a,b $\in$ G

Self-Dual RA-semigroup : An RA-semigroup which satisfies left invertive law $(ab)c = c(ba)$ *for all* a,b,c $\in$ G is called Self -Dual RA-semigroup.

Nuclear square : An RA-semigroup G is called left nuclear square if $\forall\ a,b,c, \in$ G, $a^2(bc) = (a^2b)c$. Similarly S is called right nuclear square if $\forall\ a,b,c, \in$ G, $(ab)c^2 = a(bc^2)$ and middle nuclear square if $\forall a,b,c, \in$ G, $(ab^2)c = a(b^2c)$.

Right transitive RA-smigreoup : An RA-semigroup G is called right transitive if , $ab.cb = ac$ for all $a,b, c \in$ G

Left transitive RA-semigroup : An RA-semigroup G is called left transitive if , $ab.ac = bc$ for all $a,b, c \in$ G

Cancellative RA-semigroup : Let G be an RA-semigroup. If for all $a, b ,c, \in$ G, $ab = ac \implies b =c$ then we say that G is a left cancellative RA-semigroup

Let G be an RA-semigroup. If for all $a, b ,c, \in$ G, $ba = ca \implies b =c$ then we say that G is a right cancellative RA-semigroup

Let G be an RA-semigroup. If for all $a, b ,c, \in$ G, $ab = ac \implies b =c$, and also $ba = ca \implies b = c$, then we say that G is a cancellative RA-semigroup

Let G be an RA-semigroup. If for all $a, b ,c, \in$ G, $ab = ca \implies b =c$, and also $ba = ac \implies b = c$, then we say that G is a cross-cancellative RA-semigroup

Definition : Let G be an RA semigroup and $a,b \in$ G. If there exists an element c $\in$ G such that $ab = bc$, then c is called conjugate of a by b and it is denoted by a^b . If for all elements a,b in G, a^b exists we say that G admits conjugates.

Therefore for any a and b in a cancellative semigroup G admits conjugates ,a^b is unique and $ab = ba^{b.}$

III.RA-SEMIGROUPS WITH ADDITIONAL PROPERTIES WHICH ADMITS CONJUGATES

3.1 Theorem: Let G be a left alternative RA-semigroup then, G admits conjugates and further conjugates of an element a of G by any element of G is same which is equal to a.

Proof **:** Let G be a left alternative RA-semigroup.

(i) Let 'a' be an arbitrary element of G.

By left alternativity in G we have , $(aa)b = a(ab) \quad \forall\, a, b \in$ G.

Using right invertive law on the right we get $(aa)b = b(aa) \quad \forall\, a, b \in$ G.

$\Longrightarrow$ G is commutative.

Since G is commutative for any a,b in G we have $ab = ba$

i.e., a is a conjugate of a by b.

i.e., $a^b = a \quad \forall\, a, b \in$ G.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \quad \forall\, a, b, c \in$ G.

By part (i) we have $ab = ba \Longrightarrow a^b = a$ and $ac = ca \Longrightarrow a^c = a$

thus $a^b = a^c = a \quad \forall\, a, b, c \in$ G.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a***

3.2 Theorem : Let G be a cross cancellative RA-semigroup. if G is right alternative then, G admits conjugates and further conjugates of an element a of G by any element of G is same which is equal to a.

Proof : Let G be a cross cancellative RA-semigroup. And let G be right alternative.

(i) Let 'a' be an arbitrary element of G.

Since G is right alternative we have , $(ab)b = a(bb) \quad \forall\, a, b \in$ G.

Using right invertive law on the right we get $(ab)b = b(ba)$

using cross-cancellativity in G we have $ab = ba \quad \forall\, a, b \in$ G.

$\Longrightarrow$ G is commutative.

Since G is commutative for any a,b in G we have $ab = ba$

i.e., a is a conjugate of a by b i.e., $a^b = a \quad \forall\, a, b \in$ G.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \quad \forall\, a, b, c \in$ G.

By part (i) we have $ab = ba \Longrightarrow a^b = a$ and $ac = ca \Longrightarrow a^c = a$

thus $a^b = a^c = a \quad \forall\, a, b, c \in$ G.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a***

3.3 Theorem : Let G be a left cancellative RA-semigroup. If G is right commutative then, G admits conjugates and further conjugates of an element a of G by any element of G is same which is equal to a.

Proof : Let G be a left cancellative RA-semigroup. And let G be right commutative.

(i) Let 'a' be an arbitrary element of G.

Since G is right commutative we have , $a(bc) = a(cb) \quad \forall\, a, b, c \in G$.

using left cancellativity in G we have $bc = cb \quad \forall\, b, c \in G$.

$\Longrightarrow$ G is commutative.

Since G is commutative for any a, b in G we have $ab = ba$

i.e., a is a conjugate of a by b.

i.e., $a^b = a \quad \forall\, a, b \in G$.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \quad \forall\, a, b, c \in G$.

By part (i) we have $ab = ba \Longrightarrow a^b = a$ and $ac = ca \Longrightarrow a^c = a$

thus $a^b = a^c = a \quad \forall\, a, b, c \in G$.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a***

3.4 Theorem : Let G be a right cancellative RA-semigroup. If G is left commutative then, G admits conjugates and further conjugates of an element a of G by any element of G is same which is equal to a.

Proof : Let G be a right cancellative RA-semigroup. And let G be left commutative.

(i) Let 'a' be an arbitrary element of G.

Since G is left commutative we have , $(ab)c = (ba)c \quad \forall\, a, b, c \in G$.

using right cancellativity in G we have $ab = ba \quad \forall\, a, b \in G$.

$\Longrightarrow$ G is commutative.

Since G is commutative for any a, b in G we have $ab = ba$

i.e., a is a conjugate of a by b.

i.e., $a^b = a \quad \forall\, a, b \in G$.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \quad \forall\, a, b, c \in G$.

By part (i) we have $ab = ba \Longrightarrow a^b = a$ and $ac = ca \Longrightarrow a^c = a$

thus $a^b = a^c = a \quad \forall\, a, b, c \in G$.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a***

3.5 Theorem **:** Let G be an RA-semigroup with left identity then, G admits conjugates and further conjugates of an element *a* of G by any element of G is same which is equal to *a*.

Proof : Let G be an RA-semigroup with left identity . And let ***e*** be left identity in G

(i) Let *'a'* be an arbitrary element of G.

Since ***e*** is the left identity in G, we have $ea = a \ \forall \ a \in$ G.

Now consider $a.e$

Since e is the left identity $ae= a(ee)$

by using right invertive law on the right we get , $ae = a(ee) = e(ea) = ea =a$

$\Longrightarrow$ e is the right identity

$\Longrightarrow$ e is the identity in G.

Now for any ab in G, $ab = a(be)$ (e is the right identity)

$= e(ba)$ (e is the identity)

$= (ba)$

i.e., $ab=ba \ \forall \ a,b \in$ G. $\Longrightarrow$ G is commutative.

Since G is commutative, for any a, b in G we have $ab = ba$

i.e., a is a conjugate of a by b.

i.e., $a^b =a \ \forall a, b \in$ G.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \ \forall a, b, c \in$ G.

By part (i) we have $ab = ba \Longrightarrow a^b = a$ and $ac = ca \Longrightarrow a^c = a$

thus $a^b = a^c = a \ \forall a, b, c \in$ G.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a***

3.6 Theorem: Let G be an RA-semigroup with the identity $a(bc) = (ba)c \ \forall a, b, c \in$ G then, G admits conjugates and further conjugate of an element a of G by any element of G is same which is equal to a.

Proof : Let G be an RA-semigroup with the identity $a(bc) = (ba)c \ \forall a, b, c \in$ G

then by right invertive law, $c(ba) = (ba)c \ \forall a, b, c \in$ G

$\Longrightarrow$ G is commutative

(i) Let *'a'* be an arbitrary element of G.

Since G is commutative, for any a, b in G we have $ab = ba$

i.e., a is a conjugate of a by b.

i.e., $a^b =a \ \forall a, b \in$ G.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \ \forall\ a, b, c \ \in$ G.

By part (i) we have $ab = ba \implies a^b = a$ and $ac = ca \implies a^c = a$

thus $a^b = a^c = a \ \forall\ a, b, c \in$ G.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a***

3.7 Theorem: Let G be an idempotent RA-semigroup. If G is paramedial then, G admits conjugates and further conjugate of an element a of G by any element of G is same which is equal to a.

Proof : Let G be an idempotent RA-semigroup. Then $a^2 = a \ \forall\ a, \in$ G.

Let G be paramedial then, $(ab)(cd) = (db)(ca)$

Consider $ab =(ab)^2 = (ab)(ab) = (bb)(aa) =b^2a^2 = ba$

$\implies ab = ba \ \forall\ a, b \in$ G

$\implies$ G is commutative

(i) Let 'a' be an arbitrary element of G.

Since G is commutative, for any a, b in G we have $ab = ba$

i.e., a is a conjugate of a by b.

i.e., $a^b =a \ \forall\ a, b \in$ G.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \ \forall\ a, b, c \ \in$ G.

By part (i) we have $ab = ba \implies a^b = a$ and $ac = ca \implies a^c = a$

thus $a^b = a^c = a \ \forall\ a, b, c \in$ G.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a***

3.8 Theorem: Let G be an RA-semigroup with right identity. If G is right commutative then, G admits conjugates and further conjugate of an element a of G by any element of G is same which is equal to a.

Proof : Let G be an RA-semigroup with right identity. Then $ae = a \ \forall\ a, \in$ G.

Let G be right commutative, $a(bc) = a(cb)$

Consider $ea = e(ae) = e(ea) = a(ee) =ae = a$

$\implies ea = a$

$\implies e$ is the left identity.

$\implies e$ is the identity

Consider $ab = a(be)=e(ba)=$ ba $\implies ab = ba \ \forall\ a, b \in$ G

$\implies$ G is commutative

(i) Let 'a' be an arbitrary element of G.

Since G is commutative, for any a, b in G we have $ab = ba$

i.e., a is a conjugate of a by b.

i.e., $a^b = a \ \forall\, a, b \in$ G.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \ \forall\, a, b, c \in$ G.

By part (i) we have $ab = ba \implies a^b = a$ and $ac = ca \implies a^c = a$

thus $a^b = a^c = a \ \forall\, a, b, c \in$ G.

The conjugate of an element $\boldsymbol{a}$ of G by any element of G is same which is equal to $\boldsymbol{a}$

3.9 Theorem: Let G be an RA-semigroup with right identity. If G is self-dual then, G admits conjugates and further conjugate of an element a of G by any element of G is same which is equal to a.

Proof : Let G be an RA-semigroup with right identity. Then $ae = a \ \forall\, a, \in$ G.

Let G be self-dual then, $(ab)c = (cb)a$

Consider $ea = (ee)a = (ae)e = ae = a$

$\implies ea = a$

$\implies e$ is the left identity.

$\implies e$ is the identity

Now $ab = a(be) = e(ba) =$ ba $\implies ab = ba \ \forall\, a, b \in$ G

$\implies$ G is commutative

(i) Let 'a' be an arbitrary element of G.

Since G is commutative, for any a, b in G we have $ab = ba$

i.e., a is a conjugate of a by b.

i.e., $a^b = a \ \forall\, a, b \in$ G.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \ \forall\, a, b, c \in$ G.

By part (i) we have $ab = ba \implies a^b = a$ and $ac = ca \implies a^c = a$

thus $a^b = a^c = a \ \forall\, a, b, c \in$ G.

The conjugate of an element $\boldsymbol{a}$ of G by any element of G is same which is equal to $\boldsymbol{a}$

3.10 Theorem: Let G be a cross cancellative RA-semigroup. if G is right nuclear square then, G admits conjugates and further conjugates of an element a of G by any element of G is same which is equal to a.

Proof : Let G be a cross cancellative RA-semigroup.

And let G be a nuclear square RA-semigroup.

(i) Let '*a*' be an arbitrary element of G.

Since G is left nuclear square. Then we have , $(a^2b)c = a^2(bc)$ $\forall\, a, b, c \in$ G.

Using right invertive law on the right we get $(a^2b)c = c(ba^2)$

using cross-cancellativity in G we have $a^2b = ba^2$ $\forall\, a^2, b \in$ G.

$\Longrightarrow$ G is commutative.

Since G is commutative for any *a,b* in G we have $ab = ba$

i.e., *a* is a conjugate of *a* by *b*.

i.e., $a^b = a$ $\forall\, a, b \in$ G.

Since *a* is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any *a,b,c* in G we have,

$ab = ba$ and $ac = ca$ $\forall\, a, b, c \in$ G.

By part (i) we have $ab = ba \Longrightarrow a^b = a$ and $ac = ca \Longrightarrow a^c = a$

thus $a^b = a^c = a$ $\forall\, a, b, c \in$ G.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a.***

Remark : The above theorem also holds good for right, middle nuclear square RA-semigroups.

3.11 Theorem: Let G be a left transitive RA-semigroup then, G admits conjugates and further conjugates of an element *a* of G by any element of G is same which is equal to *a.*

Proof : Let G be a left transitive RA-semigroup. Then $ab.ac = bc$ $\forall\, a, b, c \in$ G.

(i) Let '*a*' be an arbitrary element of G.

By left transitive condition in G we have , $bc = bb.bc$

$$bb.bc = bb(bb.bc) \qquad (\textit{Since } bc=bb.bc).$$

Using right invertive law on the right we get $bc = bb(bb.bc)$

$$= bc(bb.bb$$

$$= bc.bb = cb \qquad (\textit{by left transitivity in G})$$

Thus $bc = cb$ $\Longrightarrow$ G is commutative.

Since G is commutative for any *a,b* in G we have $ab = ba$

i.e., *a* is a conjugate of *a* by *b* i.e., $a^b = a$ $\forall\, a, b \in$ G.

Since *a* is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any *a,b,c* in G we have,

$ab = ba$ and $ac = ca$ $\forall\, a, b, c \in$ G.

By part (i) we have $ab = ba \Longrightarrow a^b = a$ and $ac = ca \Longrightarrow a^c = a$

thus $a^b = a^c = a$ $\forall\, a, b, c \in$ G.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a***

3.12 Theorem: Let G be right transitive RA-semigroup then, G admits conjugates and further conjugates of an element a of G by any element of G is same which is equal to a.

Proof : Let G be a right transitive RA-semigroup. Then $ab.cb = ac. \ \forall a, b, c \in G$.

(i) Let 'a' be an arbitrary element of G.

By left transitive condition in G we have , $ac = ac.aa = (a.ac)(a.aa)$

Using right invertive law on the right we get $ac = (c.aa)(a.aa)$ *(by right transitivity in G)*

$= ca$ *(by right transitivity in G)*

Thus $bc = cb \implies$ G is commutative.

Since G is commutative for any a,b in G we have $ab = ba$

i.e., a is a conjugate of a by b.

i.e., $a^b = a \ \forall a, b \in G$.

Since a is the arbitrary element in G, G admits conjugates.

(ii) Since G is commutative, then for any a,b,c in G we have,

$ab = ba$ and $ac = ca \ \forall a, b, c \in G$.

By part (i) we have $ab = ba \implies a^b = a$ and $ac = ca \implies a^c = a$

thus $a^b = a^c = a \ \forall a, b, c \in G$.

The conjugate of an element ***a*** of G by any element of G is same which is equal to ***a***

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